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Final

MATH 534A
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Do 3 of the following 4 problems. Do them all for extra credit.

1. Define $\varphi(A) = \det(A) \in \mathbb{R}$ for $A \in \mathrm{GL}(n, \mathbb{R})$. This problem is devoted to the proof of the identity,

$$\varphi_{*,A}(X) = \frac{d}{dt}\det(A + tX)|_{t=0} = \det(A)\mathrm{Tr}(A^{-1}X)$$

and the identity,

$$\det(e^{tX}) = e^{t\mathrm{Tr}(X)}$$

Recall that $\mathrm{Tr}(X) = \sum_k X_{k,k}$, the sum of the diagonal entries of the matrix X.

(a) Show by a direct calculation that at the identitiy, I,

$$\varphi_{*,I}(X) = \mathrm{Tr}(X).$$

Use this and the multiplicative property of determinants to evaluate $\phi_{*,A}(X)$.

Solution. For the first part, use Jacobi's formula, which states that

$$\frac{d\det(A)}{dt} = \mathrm{Tr}\big(\mathrm{Adj}(A)\frac{dA}{dt}\big).$$

Here, Adj is the classical adjoint. For the current problem, $A = I + tX$. Then, it follows that

$$\begin{aligned}\varphi_{*,I}(X) = \frac{d\det(I + tX)}{dt}\Big|_{t=0} &= \mathrm{Tr}\big(\mathrm{Adj}(I + tX)\frac{d(I + tX)}{dt}\big)\Big|_{t=0} \\ &= \mathrm{Tr}(\mathrm{Adj}(I)X).\end{aligned}$$

But, the (i, i) subminors that make up the matrix $\mathrm{Adj}(I)$ are all $+1$, and therefore action by the classical adjoint preserves trace. Therefore, conclude that

$$\begin{aligned}\varphi_{*,I}(X) &= \mathrm{Tr}(\mathrm{Adj}(I)X) \\ &= \mathrm{Tr}(X).\end{aligned}$$

Now, consider the above for $\varphi_{*,A}(X)$. First, rewrite $A + tX = A(I + tA^{-1}X)$. Now, use the multiplicative property of determinants and we have that

$$\det(A + tX) = \det(A(I + tA^{-1}X) = \det(A)\det(I + tA^{-1}X).$$

Now, take the time derivative and evaluate at $t = 0$. The $\det(A)$ is not time-dependent and therefore can be factored out from under the derivative,

$$\begin{aligned}\varphi_{*,A} = \frac{d\det(A + tX)}{dt}\Big|_{t=0} &= \frac{d\det(A)\det(I + tA^{-1}X)}{dt}\Big|_{t=0} \\ &= \det(A)\frac{d\det(I + tA^{-1}X)}{dt}\Big|_{t=0} \\ &= \det(A)\mathrm{Tr}(A^{-1}X) \text{ for the same reason as above}\end{aligned}$$

(b) Any continuous map $\mathbb{R} \ni t \to f(t) \in \mathbb{R}$ which satisfies $f(s+t) = f(s)f(t)$ and $f(0) = 1$ must be of the form $f(t) = e^{ta}$ for some constant a. Use this and part (a) to show that,

$$\det(e^{tX}) = e^{t\mathrm{Tr}(X)}.$$

Note: $t \to e^{tX}$ and $t \to I + tX$ have the same tangent at $t = 0$.

Solution. Let $f_X(t) = \det(e^{tX})$. Clearly this is a continuous map from $\mathbb{R} \to \mathbb{R}$ and note that

$$f_X(s+t) = \det(e^{(s+t)X}) = \det(e^{sX}e^{tX}) = \det(e^{sX})\det(e^{tX}) = f_X(s)f_X(t).$$

Also, as will become handy later, (see hint) $e^{tX} = I + tX + t^2/2X^2 + \dots$. At $t = 0$, $e^{0X} = I$, and so $f_X(0) = 1$. Now, we can use the relation given, i.e. that

$$f_X(t) = \det(e^{tX}) = e^{ta}$$

for some constant a.

Now, differentiate and evaluate at $t = 0$ to find a.

$$\begin{aligned} a = \frac{de^{ta}}{dt}\Big|_{t=0} &= \frac{d\det(e^{tX})}{dt}\Big|_{t=0} \\ &= \frac{d\det(I + tX + \frac{t^2}{2}X^2 + \dots)}{dt}\Big|_{t=0}. \end{aligned}$$

All powers higher than order one go away at $t = 0$, therefore we can look at the simpler, equivalent problem

$$\begin{aligned} \frac{d\det(I + tX + \frac{t^2}{2}X^2 + \dots)}{dt}\Big|_{t=0} &= \frac{d\det(I + tX)}{dt}\Big|_{t=0} \\ &= \mathrm{Tr}(X) \text{ from above} \end{aligned}$$

Therefore, $a = \mathrm{Tr}(X)$ and

$$\det(e^{tX}) = e^{t\mathrm{Tr}(X)}.$$

□

2. The map

$$(x, y) \to z = x + iy$$

identifies $\mathbb{R}^2$ with the complex plane, C in the usual fashion. Define

$$p_n(z) = z^n.$$

With this standard identification we can think of p_n as a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Show that with this way of identifying $\mathbb{R}^2$ with C, we have,

$$p_{n,*,z}(v) = nz^{n-1}v$$

where $v \in T_z\mathbb{R}^2 \simeq C$ and the product on the right is just multiplication of complex numbers.

Solution. Consider the map $t \mapsto c_z^n(t)$ defined by $c_z^n(t) = (z+tv)^n$, for some $v = (v_1, v_2) \in T_z\mathbb{C}$. This map passes through z at $t=0$ and therefore $(c_z^n)'(0) \in T_z\mathbb{C}$. This tangent vector is precisely $p_{n,*,z}$, and can be evaluated as

$$\begin{aligned} p_{n,*,z} = \frac{d}{dt}(z+tv)^n|_{t=0} &= n(z+tv)^{n-1}v|_{t=0} \\ &= nz^{n-1}v \end{aligned}$$

(a) Use this to show that if $p(z)$ is a polynomial in z, then

$$p_{*,z}(v) = p'(z)v,$$

where $v \in T_z\mathbb{R}^2 \simeq C$ and $p'(z)$ is the usual derivative of the polynomial p.

Solution. This follows directly from above and the linearity and scalar property of the derivative, that is

$$\begin{aligned} \frac{d\sum_{i=1}^n a_i(z+tv)^i}{dt}\Big|_{t=0} &= \sum_{i=1}^n a_i \frac{d(z+tv)^i}{dt}\Big|_{t=0} \\ &= \sum_{i=0}^{n-1} a_i i z^{i-1} v \\ &= p'(z)v. \end{aligned}$$

Therefore, $p_{*,z}(v) = p'(z)v$.

(b) Suppose that

$$p(z) = z^n + p_{n-1}z^{n-1} + \cdots + p_0$$

where z and p_j for $j = 0, \ldots, n-1$ are complex numbers. Use the implicit function theorem and part (a) to show that: If $p(z)$ has *distinct* roots $z_1, z_2, \ldots, z_n$ then there exist C^∞ functions $q \to z_j(q)$ defined for $q \in C^n \simeq \mathbb{R}^{2n}$ in sufficiently small neighborhood of

$$p = (p_0, p_1, \ldots, p_{n-1})$$

So that

$$z_j(q)^n + q_{n-1}z_j(q)^{n-1} + \cdots + q_0 = 0,$$

and $z_j(p) = z_j$. This shows that the roots of a polynomial are locally C^∞ functions of the coefficients of the polynomial whenever the roots are distinct. Hint: Calculate the rank of the derivative map,

$$(z, q) \to z^n + q_{n-1}z^{n-1} + \cdots + q_0$$

at the points (z_j, p). Use the results of 2(a) to do this calculation using complex variables (even though you use the real implicit function theorem).

Solution. Consider the map,

$$f(z,q) = z^n + q_{n-1}z^{n-1} + \cdots + q_0.$$

When evaluating $f(z_j, p)$, the result is 0 for all $1 \le j \le n$. Consider coordinates $(\mathbf{x}, \mathbf{y})$. Fix the point

$$(\mathbf{x}', \mathbf{y}') = (z_1, \ldots, z_n, p_0, \ldots, p_{n-1}).$$

The derivative map (considering $f_i = f$ with z_i as the z variable), is then

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1'} & \cdots & \frac{\partial f_1}{\partial x_n'} & \frac{\partial f_1}{\partial y_1'} & \cdots & \frac{\partial f_1}{\partial y_n'} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1'} & \cdots & \frac{\partial f_n}{\partial x_n'} & \frac{\partial f_n}{\partial y_1'} & \cdots & \frac{\partial f_n}{\partial y_n'} \end{pmatrix}$$

$$= \begin{pmatrix} nz_1^{n-1} + \sum_{i=1}^{n-1} iq_i z_1^{i-1} & \cdots & 0 & 1 & \cdots & (n-1)p_{n-1}z_1^{n-2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & nz_n^{n-1} + \sum_{i=1}^{n-1} iq_i z_n^{i-1} & 1 & \cdots & (n-1)p_{n-1}z_n^{n-2} \end{pmatrix}.$$

For the Implicit Function Theorem application, we only care about the matrix of partial derivatives wrt the $\mathbf{y}$-coordinates. That is

$$Y = \begin{pmatrix} 1 & \cdots & (n-1)p_{n-1}z_1^{n-2} \\ \vdots & \vdots & \vdots \\ 1 & \cdots & (n-1)p_{n-1}z_n^{n-2} \end{pmatrix}.$$

This resembles the form of a Vandermonde matrix very closely; the determinant of which is non-zero for distinct z_j, as given in this problem. Therefore, we can conclude that Y is invertible.

By the Inverse Function Theorem, there are open sets $U \ni \mathbf{x}'$ and $V \ni \mathbf{y}'$ $(U, V \in \mathbb{C}^n)$ and a unique C^∞ function $g : U \to V$ s.t.

$$\{(\mathbf{x}, g(\mathbf{x}))\} = \{(\mathbf{x}, \mathbf{y}) | f(\mathbf{x}, \mathbf{y}) = \mathbf{0}\} \cap (U \times V).$$

Therefore, the roots of a polynomial are locally C^∞ functions of the coefficients of the polynomial whenever the roots are distinct.

3. Let $u_1 = \frac{x}{1-z}$ and $u_2 = \frac{y}{1-z}$ be stereographic coordinates for a point $(x, y, z) \in S^2$, the two sphere. let $p : \mathbb{R}^2 \to S^2$ given by,

$$p(u) = \frac{(2u_1, 2u_2, |u|^2 - 1)}{|u|^2 + 1} \in S^2,$$

denote the inverse (here $|u|^2 = u_1^2 + u_2^2$). Define a vector field, $V(u)$, on $\mathbb{R}^2$ by,

$$(u_1^2 - u_2^2)\frac{\partial}{\partial u_1} + 2u_1u_2\frac{\partial}{\partial u_2}$$

(a) Show that the vector field p_*V defined on $S^2 \setminus \{n\}$ where $n = (0, 0, 1)$ is the north pole extends to a smooth vector field on all of S^2. Hint: the change of coordinates to stereographic coordinates $v \in \mathbb{R}^2$ projected from the south pole is $v = \frac{u}{|u|^2}$.

Solution.

(b) Let φ_t denote the one parameter flow associated with the vector field p_*V on S^2. Find a formula for $\varphi_t(p)$ for $p \in S^2$. What are the limits for $\varphi_t(p)$ as $t \to \pm\infty$?

Solution.

4. Suppose that M is a two dimensional submanifold of $\mathbb{R}^3$ given as the graph $z = f(x, y)$ of a smooth function f on $\mathbb{R}^2$. Define the projection,

$$M \ni (x, y, z) \to p(x, y, z) = (x, y).$$

The induced volume form Ω obtained by contracting the unit normal vector field on M with $dx \wedge dy \wedge dz$ must be some function g on M times the two form $p^*(dx \wedge dy)$. Find the function g.

Solution. By definition,

$$p^*(dx \wedge dy) = (p^*dx) \wedge (p^*dy)$$

and

$$\begin{aligned} p^*dx &= d(p^*x) \\ &= d(x \circ p) \\ &= f_x dx \\ p^*dy &= d(p^*y) \\ &= d(y \circ p) \\ &= f_y dy \end{aligned}$$

So, $p^*(dx \wedge dy) = (f_x dx) \wedge (f_y dy)$.

Also, from the relation $z = f(x, y)$, $dz = df = f_x dx + f_y dy$. Consider $F(x, y, z) = f(x, y) - z = 0$. By the RLST, $\{(x, y, z) | F(x, y, z) = 0\}$ is a regular submanifold of $\mathbb{R}^3$. Note: $F_z = -1, F_x = f_x$, and $F_y = f_y$. Then,

$$\begin{aligned} -(dx \wedge dy) &= \frac{dy \wedge (F_x dx + F_y dy)}{F_x} = \frac{(F_x dx + F_y dy) \wedge dx}{F_y} \\ &= \frac{dy \wedge dz}{F_x} = \frac{dz \wedge dx}{F_y} \\ \Longrightarrow \quad \frac{dx \wedge dy}{F_z} &= \frac{dy \wedge dz}{F_x} = \frac{dz \wedge dx}{F_y} \end{aligned}$$

The above relation shows that there is a nowhere vanishing two-form on M, and that it must be some function times $dx \wedge dy$.

From a few paragraphs back, $p^*(dx \wedge dy) = (f_x dx) \wedge (f_y dy)$. Now, need to take into account the height. Therefore, $g = z$, and $\omega = zp^*(dx \wedge dy)$.